[4] Wang, C. T., "Principle and application of complementary energy method for thin homogeneous and sandwich plates and shells with finite deflections," NACA TN 2620 (1952).

[5] Hoff, N. J., "Bending and buckling of rectangular sandwich plates," NACA TN 2225 (1950).

[6] Eringen, A. C., "Bending and buckling of sandwich plates," *Proceedings of the First U. S. National Congress of Applied Mechanics* (American Society of Mechanical Engineers, New York, 1951), pp. 381–390.

[7] Alwan, A. M., "Large and small deflection analyses applied to rectangular sandwich plates with orthotropic cores," Ph.D. Thesis, Dept. of Engineering Mechanics, Univ. of Wisconsin, Madison, Wis. (June 1963).

Stress Functions for the Axisymmetric, Orthotropic, Elasticity Equations

L. R. Herrmann*

Aerojet General Corporation, Sacramento, Calif.

I. Introduction

SOUTHWELL expressed the axisymmetric elastic field equations for an isotropic body in terms of two stress functions.[1] The Southwell stress functions have found considerable application in the structural analyses of solid rocket motors.[2–4] These analyses, however, are only applicable to motors constructed of isotropic propellants, and, thus, some recent proposed motor designs that make use of orthotropic propellants give rise to new unsolved problems. So that similar analysis techniques may be applied to orthotropic grains as were applied to isotropic grains, it is desirable to express the problem in terms of stress functions of a nature similar to the Southwell functions. In particular, such functions should reduce to those of Southwell's for the particular case of isotropy.

II. Theory

The linear elastic equations expressed in cylindrical coordinates for a cylindrical orthotropic axisymmetric body subjected to axisymmetric loads are

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + F_r = 0 \tag{1}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r} + F_z = 0 \tag{2}$$

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} = S_{11}\tau_{rr} + S_{12}\tau_{\theta\theta} + S_{13}\tau_{zz} + e_1 \tag{3}$$

$$\epsilon_{\theta\theta} = u_r/r = S_{12}\tau_{rr} + S_{22}\tau_{\theta\theta} + S_{23}\tau_{zz} + e_2 \tag{4}$$

$$\epsilon_{zz} = \partial u_z/\partial z = S_{13}\tau_{rr} + S_{23}\tau_{\theta\theta} + S_{33}\tau_{zz} + e_3 \tag{5}$$

$$\epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = \frac{S_{44}}{2}\tau_{rz} \tag{6}$$

Equations (1) and (2) are the equilibrium equations, and Eqs. (3–6) are the stress (τ_{ij}) strain (ϵ_{ij}) law for a cylindrical orthotropic material[5] (displacement components are denoted by u_i and the orthotropic elastic constants by S_{ij}). The free expansion in the i direction caused by a temperature change $\Delta T = T - T_0$ is given by e_i, i.e.,

$$e_i = \int_{T_0}^{T} \alpha_i dT'$$

The instantaneous coefficient of linear expansion is denoted by α_i, i.e., $\alpha_i = (1/L_i)(dL_i/dT)$.

A stress function $\psi(r,z)$ is defined by the following equation:

$$\tau_{zz} = -(1/r)(\partial\psi/\partial r) \tag{7}$$

Received March 23, 1964; revision received May 18, 1964.

* Senior Research Engineer.

A function $f_z(r,z)$ related to the axial body force $F_z(r,z)$ is defined by the following equation:

$$F_z = (1/r)(\partial f_z/\partial r) \tag{8}$$

Introducing Eqs. (7) and (8) into Eq. (2) gives

$$\frac{\partial}{\partial r}(r\tau_{rz}) - \frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial z}\right) + \frac{\partial f_z}{\partial r} = 0 \tag{9}$$

Integration of Eq. (9) yields (it may be simply shown that the function of integration may be set equal to zero without loss of generality)

$$\tau_{rz} = \frac{1}{r}\frac{\partial \psi}{\partial z} - \frac{1}{r}f_z \tag{10}$$

Introducing Eq. (7) into Eqs. (3) and (4) and solving for τ_{rr} and $\tau_{\theta\theta}$, the following expressions are found:

$$\tau_{\theta\theta} = \frac{S_{12}\dfrac{\partial u_r}{\partial r} - S_{11}\dfrac{u_r}{r} + (S_{12}S_{13} - S_{11}S_{23})\dfrac{1}{r}\dfrac{\partial \psi}{\partial r} - S_{12}e_1 + S_{11}e_2}{(S_{12})^2 - S_{11}S_{22}} \tag{11}$$

$$\tau_{rr} = \frac{S_{12}\dfrac{u_r}{r} - S_{22}\dfrac{\partial u_r}{\partial r} + (S_{23}S_{12} - S_{22}S_{13})\dfrac{1}{r}\dfrac{\partial \psi}{\partial r} + S_{22}e_1 - S_{12}e_2}{(S_{12})^2 - S_{11}S_{22}} \tag{12}$$

The problem could now be expressed in terms of the stress function ψ and the displacement component u_r; however, in order to obtain a formulation that reduces to the Southwell formulation (for an isotropic material), it is necessary to express u_r in terms of ψ and a new function φ†:

$$u_r = \frac{S_{11}S_{22} - (S_{12})^2}{S_{22}}\frac{\varphi}{r} + \frac{S_{22}(S_{44} + S_{13} - S_{11}) + S_{12}(S_{12} - S_{23})}{S_{22}}\frac{\psi}{r} \tag{13}$$

Introducing Eq. (13) into Eqs. (11) and (12) and making the definitions

$$d_1 = (S_{12})^2 - S_{11}S_{22} \tag{14}$$

$$d_2 = -\left[1 + \frac{1}{d_1}(S_{22}S_{44} + S_{22}S_{13} - S_{12}S_{23})\right] \tag{15}$$

$$d_3 = -\left[1 + \frac{1}{d_1}(S_{22}S_{44} + 2S_{22}S_{13} - 2S_{12}S_{23})\right] \tag{16}$$

$$d_4 = -\left[\frac{S_{12}}{S_{22}}\left(1 - \frac{S_{12}S_{23}}{d_1}\right) + \frac{S_{12}(S_{44} + 2S_{13}) - S_{11}S_{23}}{d_1}\right] \tag{17}$$

the expressions for τ_{rr} and $\tau_{\theta\theta}$ are found to be

$$\tau_{rr} = \frac{1}{r}\left[\frac{\partial\varphi}{\partial r} + d_3\frac{\partial\psi}{\partial r}\right] - \left(1 + \frac{S_{12}}{S_{22}}\right)\frac{1}{r^2}[d_2\psi + \varphi] + \frac{S_{22}e_1 - S_{12}e_2}{d_1} \tag{18}$$

$$\tau_{\theta\theta} = -\frac{1}{r}\left[\frac{S_{12}}{S_{22}}\frac{\partial\varphi}{\partial r} + d_4\frac{\partial\psi}{\partial r}\right] + \frac{(S_{12} + S_{11})}{S_{22}}\frac{1}{r^2}[d_2\psi + \varphi] + \frac{S_{11}e_2 - S_{12}e_1}{d_1} \tag{19}$$

† There are many different relationships between u_r, φ, and ψ which would lead to the desired results. The particular one used herein was selected in an attempt to simplify the resulting equations as much as possible.

One of the governing equations for the two functions φ and ψ is obtained by substituting Eqs. (10, 18, and 19) into Eq. (1):

$$\frac{\partial^2 \varphi}{\partial r^2} - \frac{1}{r}\frac{\partial \varphi}{\partial r} + \left(1 - \frac{S_{11}}{S_{22}}\right)\frac{\varphi}{r^2} + d_3 \frac{\partial^2 \psi}{\partial r^2} + \left[d_4 - d_2\left(1 + \frac{S_{12}}{S_{22}}\right)\right]\frac{1}{r}\frac{\partial \psi}{\partial r} + d_2\left(1 - \frac{S_{11}}{S_{22}}\right)\frac{\psi}{r^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{S_{22}}{d_1} r \frac{\partial e_1}{\partial r} + \frac{(S_{22} + S_{12})}{d_1} e_1 - \frac{S_{12}}{d_1} r \frac{\partial e_2}{\partial r} - \frac{(S_{12} + S_{11})}{d_1} e_2 - \frac{\partial f_z}{\partial z} + rF_r = 0 \quad (20)$$

The other equation relating φ and ψ is obtained by differentiation of Eq. (5) with respect to r and Eq. (6) with respect to z and eliminating the displacement w from the resulting equations [Eqs. (7, 10, 13, 18, and 19) are utilized to simplify the resulting expression]:

$$\left(S_{13} - \frac{S_{23}S_{12}}{S_{22}}\right)\frac{\partial^2 \varphi}{\partial r^2} + \left[\frac{S_{23}}{S_{22}}(2S_{12} + S_{11}) - S_{13}\left(2 + \frac{S_{12}}{S_{22}}\right)\right]\frac{1}{r}\frac{\partial \varphi}{\partial r} + 2\left[S_{13}\left(1 + \frac{S_{12}}{S_{22}}\right) - S_{23}\frac{S_{12} + S_{11}}{S_{22}}\right]\frac{\varphi}{r^2} - \frac{d_1}{S_{22}}\frac{\partial^2 \varphi}{\partial z^2} + (S_{13}d_3 - S_{23}d_4 - S_{33})\frac{\partial^2 \psi}{\partial r^2} + \left[\frac{S_{23}}{S_{22}}(S_{12} + S_{11})\, d_2 + S_{33} + S_{23}d_4 - S_{13}d_3 - S_{13}\left(1 + \frac{S_{12}}{S_{22}}\right) d_2\right]\frac{1}{r}\frac{\partial \psi}{\partial r} + 2d_2\left[S_{13}\left(1 + \frac{S_{12}}{S_{22}}\right) - S_{23}\frac{S_{12} + S_{11}}{S_{22}}\right]\frac{\psi}{r^2} + \frac{d_1 - S_{12}S_{23} + S_{22}S_{13}}{S_{22}}\frac{\partial^2 \psi}{\partial z^2} + \frac{(S_{13}S_{22} - S_{23}S_{12})}{d_1} r \frac{\partial e_1}{\partial r} + \frac{(S_{23}S_{11} - S_{13}S_{12})}{d_1} r \frac{\partial e_2}{\partial r} + r\frac{\partial e_3}{\partial r} + S_{44}\frac{\partial f_z}{\partial z} = 0 \quad (21)$$

Equations (20) and (21) can be solved simultaneously for φ and ψ; however, to express the governing equations in a form analogous to the Southwell equations, they are cast into a slightly different form. The following additional definitions shall be employed in the final equations:

$$d_5 = (1/d_1)[(S_{23})^2 - S_{33}S_{22}]$$

$$d_6 = -\left\{\frac{[S_{23}(S_{12} + S_{22}) - S_{13}S_{22}]d_4 - S_{13}S_{22}d_3 + S_{33}S_{22}}{d_1} - \frac{S_{23}}{S_{22}} d_2\right\}$$

$$d_7 = \left[\frac{S_{23}(S_{12} + S_{11}) - S_{13}(S_{22} + S_{12})}{d_1}\right]$$

$$d_8 = \frac{1}{d_1}\left[\left(1 + \frac{S_{11}}{S_{22}}\right)(S_{13}S_{22} - S_{23}S_{12}) + 2(S_{13}S_{12} - S_{23}S_{11})\right]$$

$$d_9 = \frac{d_2}{d_1}\left[S_{13}(S_{11} + S_{22} + 2\,S_{12}) - S_{23}\left(S_{12} + 2S_{11} + \frac{S_{11}S_{12}}{S_{22}}\right)\right]$$

$$d_{10} = (1/d_1)(S_{23}S_{12} - S_{13}S_{22})$$

$$d_{11} = (1/d_1)(S_{44}S_{22} - S_{23}S_{12} + S_{13}S_{22})$$

$$d_{12} = 1 - (S_{11}/S_{22}) - d_8$$

$$d_{13} = d_4 - d_2\left(1 + \frac{S_{12}}{S_{22}}\right) + d_6$$

$$d_{14} = d_2\left(1 - \frac{S_{11}}{S_{22}}\right) - d_9$$

$$d_{15} = (1/d_1)(S_{22} + S_{12})(1 - d_{10})$$

$$d_{16} = -(1/d_1)(S_{12} + S_{11})(1 - d_{10})$$

To cast the governing equations into the desired form, Eq. (20) is multiplied by $(S_{23}S_{12} - S_{13}S_{22})/S_{22}$ and added to Eq. (21):

$$d_5\frac{\partial^2 \psi}{\partial r^2} - d_6\frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \varphi}{\partial z^2} + d_7\frac{1}{r}\frac{\partial \varphi}{\partial r} + d_8\frac{\varphi}{r^2} + d_9\frac{\psi}{r^2} - \frac{S_{23}}{d_1} r\frac{\partial e_2}{\partial r} + \frac{d_{10}}{d_1}[(S_{22} + S_{12})e_1 - (S_{12} + S_{11})e_2] + \frac{S_{22}}{d_1} r \frac{\partial e_3}{\partial r} + d_{11}\frac{\partial f_z}{\partial z} + d_{10}\, rF_r = 0 \quad (22)$$

The second governing equation is obtained by subtracting the foregoing equation from Eq. (20):

$$\frac{\partial^2 \varphi}{\partial r^2} - (1 + d_7)\frac{1}{r}\frac{\partial \varphi}{\partial r} + d_{12}\frac{\varphi}{r^2} + \frac{\partial^2 \varphi}{\partial z^2} + (d_3 - d_5)\frac{\partial^2 \psi}{\partial r^2} + d_{13}\frac{1}{r}\frac{\partial \psi}{\partial r} + d_{14}\frac{\psi}{r^2} + \frac{S_{22}}{d_1} r \frac{\partial e_1}{\partial r} + \frac{S_{23} - S_{12}}{d_1} r\frac{\partial e_2}{\partial r} + d_{15}\, e_1 + d_{16}e_2 + (1 - d_{10})rF_r - \frac{S_{22}}{d_1} r\frac{\partial e_3}{\partial r} - (1 + d_{11})\frac{\partial f_z}{\partial z} = 0 \quad (23)$$

Thus, the linear elastic behavior of an axisymmetric body constructed of cylindrical orthotropic material and subjected to axisymmetric loads is governed by Eqs. (22) and (23). The stresses are given by Eqs. (7, 10, 18, and 19). The radial displacement is given by Eq. (13).

For an isotropic material (E and ν are the elastic constants) Eqs. (7, 10, 13, 18, 19, 22, and 23) become

$$\frac{\partial^2 \varphi}{\partial r^2} - \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} + \frac{1}{1 - \nu}\left[\frac{\partial f_z}{\partial z} + rF_r\right] = 0$$

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \varphi}{\partial z^2} - \frac{1}{1 - \nu}\left[(2 - \nu)\frac{\partial f_z}{\partial z} + \nu r F_r + \alpha E r \frac{\partial \Delta T}{\partial r}\right] = 0$$

$$\tau_{rz} = \frac{1}{r}\frac{\partial \psi}{\partial z} - \frac{1}{r} f_z$$

$$\tau_{zz} = -\frac{1}{r}\frac{\partial \psi}{\partial r}$$

$$\tau_{rr} = \frac{1}{r}\left[\frac{\partial \varphi}{\partial r} + \frac{\partial \psi}{\partial r}\right] - \frac{1}{r^2}[\psi + (1 - \nu)\varphi] - \frac{E\alpha\Delta T}{1 - \nu}$$

$$\tau_{\theta\theta} = \frac{\nu}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}[\psi + (1 - \nu)\varphi] - \frac{E\alpha\Delta T}{1 - \nu}$$

$$u_r = \frac{(1 - \nu^2)}{E}\frac{\varphi}{r} + \frac{(1 + \nu)}{E}\frac{\psi}{r}$$

These equations (in the absence of temperature and body force effects) are the same as given by Southwell for an isotropic material.[1]

III. Conclusions

The governing equations for the axisymmetric deformation of an orthotropic material were expressed in terms of two stress functions. The stress functions were selected such that, for the special case of an isotropic material, the functions are identical to the Southwell stress functions. The orthotropy considerably complicates the governing equations; however, it is felt that this should offer no complications for numerical solution methods.

References

[1] Allen, D. N., *Relaxation Methods in Engineering and Science* (McGraw-Hill Book Co., Inc., New York, 1954), p. 135.

[2] Messner, A. M., "Propellant grain stress analysis," Proceedings of the Seventh JANAF-ARPA-NASA Solid Propellant Group Meeting (May 1961), Vol. 2, p. 149.

[3] Parr, C. H., "End effects due to shrinkage in propellant grains," 20th Meeting Bulletin of the JANAF-ARPA-NASA Panel on Physical Properties of Solid Propellants (November 1961), pp. 71–80.

[4] Andrews, W. G. and Stornelli, A., "A digital computer program for analyzing structural problems in very large solid boosters," Bulletin of the 2nd Meeting Interagency Chemical Rocket Propulsion Group on Mechanical Behavior (November 1963), pp. 401–409.

[5] Hearmon, R. F. S., *An Introduction to Applied Anisotropic Elasticity* (Oxford University Press, London, 1961), pp. 17–26.

Compact Formal Analysis of Beam Columns

J. E. Brock*

U. S. Naval Postgraduate School, Monterey, Calif.

A RECENT note[1] by Urry considers the use of what he calls Macauley's brackets in the treatment of beam-column problems. It is the purpose of the present note to generalize this treatment. The notation we will use

$$\left.\begin{aligned}\{x-a\}^n &= (x-a)^n \text{ if } x \geqq a\\ &= 0 \qquad\quad\; \text{if } x < a\end{aligned}\right\} n \geqq 0$$

is that of Brock and Newton[2]; cf. a recent note by Weissenburger which discusses history and compares notations. We will also use the following additional notation:

$$\{x - a\} = \{x - a\}^1$$

That is, the unit exponent will not be exhibited. Furthermore, suppose that $f(z)$ is a given function of the variable z; we will use the notation

$$\begin{aligned}f[\alpha\{x-a\}] &= f[\alpha(x-a)] \text{ if } x \geqq a\\ &= f(0) \qquad\qquad \text{if } x < a\end{aligned}$$

Thus, for example,

$$\begin{aligned}\cos[\alpha\{x-a\}] &= \cos[\alpha(x-a)] \text{ if } x \geqq a\\ &= 1 \qquad\qquad\qquad \text{if } x < a\end{aligned}$$

In the analysis, we first consider the following functions that result from truncating the Maclaurin series for sine and cosine:

$$\text{tru}_n z = \frac{z^n}{\Gamma(n+1)} - \frac{z^{n+2}}{\Gamma(n+3)} + \frac{z^{n+4}}{\Gamma(n+5)} - \cdots$$

In most practical applications, n is a positive integer so that the gamma functions may be replaced by factorials, viz.,

$$\Gamma(n + 1) = n! \text{ (for integer } n)$$

However, the more general form may be useful in certain cases. It is easily established that

$$d(\text{tru}_n z)/dz = \text{tru}_{n-1} z \qquad n \neq 0$$

$$d^2(\text{tru}_n z)/dz^2 = \text{tru}_{n-2} z = [z^{n-2}/\Gamma(n-1)] - \text{tru}_n z \qquad n \neq 0,1$$

The first several tru-functions for integer index are:

$$\begin{aligned}\text{tru}_0 z &= \cos z\\ \text{tru}_1 z &= \sin z\\ \text{tru}_2 z &= 1 - \cos z\\ \text{tru}_3 z &= z - \sin z\\ \text{tru}_4 z &= \cos z - (1 - z^2/2!)\\ \text{tru}_5 z &= \sin z - (z - z^3/3!), \ldots\end{aligned}$$

Now consider the problem of a laterally and axially loaded beam of uniform flexural rigidity EI and length L. Let M^* denote the bending moment that would be acting if the axial load were absent. One may write

$$M^* = \Sigma c_k\{x - a_k\}^{n_k}$$

Some of the coefficients c_k may represent redundant reactions, constraining moments, etc., in the case of statically indeterminate beams. Also, there may exist relations involving M^* and/or its derivative; for example, $M^* = 0$ at a hinged support. In the case of statically indeterminate beam columns, there will be additional conditions on deflection and/or slope which, later, will provide sufficient conditions for the determination of all the c_k and the integration constants A and B introduced in what follows.

Because of the additional compressive axial load P, there is an additional contribution, $-Py$, to the bending moment; here y denotes the deflection. Thus, we are concerned with the differential relation

$$(d^2y/dx^2) + \alpha^2 y = M^*/EI \qquad \alpha^2 = P/EI$$

It may be easily verified by direct substitution that the function

$$y = A\cos(\alpha x) + B\sin(\alpha x) + \Sigma\frac{c_k(n_k)!}{P\alpha^{n_k}}\text{tru}_{n_k+2}[\alpha\{x - a_k\}]$$

satisfies this differential equation if M^* has the form given earlier. This may be considered to be the formal generalization of the use of the curly-bracket symbol (the name we prefer to use in order to prevent confusion of nomenclature; cf. Weissenburger[3]) in the beam-column problem. If the axial load is tensile, P is to be considered as negative and α becomes imaginary; however, it should be clear how algebraic signs may be manipulated and hyperbolic functions may be introduced so as to obtain convenient expressions.

It may be of interest to consider in somewhat greater detail the important case of the pin-ended statically determinate beam column. In the general case, the moment M^* due to lateral loads alone is

$$M^* = \Sigma c_k[xb_k^{n_k} - L\{x - a_k\}^{n_k}]$$

where the corresponding loading starts at $x = a_k = L - b_k$, the coefficient c_k has the dimension: (force $\times$ length$^{-n_k}$), and the exponents have the following significance: 0 $\sim$ concentrated clockwise moment, 1 $\sim$ concentrated downward load, 2 $\sim$ uniformly distributed downward load, and, for n greater than 2, $n \sim$ downward distributed load of the form $w_0\{x - a_k\}^{n-2}$. Furthermore, the condition of zero deflection at the ends permits evaluating the constants A and B, and the complete solution may be written

$$y = \Sigma\frac{Lc_k}{P}\left[b_k^{n_k}\left(\frac{x}{L} - \frac{\sin\alpha x}{\sin\alpha L}\right) - \frac{(n_k)!}{\alpha^{n_k}}\left(\text{tru}_{n_k+2}[\alpha\{x - a_k\}] - \frac{\sin\alpha x}{\sin\alpha L}\text{tru}_{n_k+2}(\alpha b_k)\right)\right]$$

As an example, if there is a single downward concentrated load F at $x = a$, we have

$$y = \frac{Fbx}{PL} - \frac{F\sin(\alpha b)\cdot\sin(\alpha x)}{P\alpha\sin(\alpha L)} - \left(\frac{F}{P\alpha}\right)(\alpha\{x - a\} - \sin[\alpha\{x-a\}])$$

Received March 27, 1964.

* Professor of Mechanical Engineering.